GAMMA-RAY PRODUCTION FROM THE DECAY OF SECONDARY PARTICLES PRODUCED IN COSMIC-RAY INTERACTIONS

1-1 INTRODUCTION

The importance of studying cosmic γ-radiation has been recognized for some time, and for fundamental reasons. These energetic photons are the direct product of interactions which are basic to high-energy astrophysics and cosmic-ray physics. Gamma-ray-producing interactions involve the basic constituents of the universe: cosmic rays, interstellar and intergalactic gas, cosmic radio waves, the universal blackbody radiation, starlight, and cosmic magnetic fields. In addition, γ-ray observations allow us to test for the presence of significant amounts of antimatter in the universe.

Gamma rays suffer negligible absorption in most cases of astrophysical interest and they travel in straight lines from their sources. In this they differ from cosmic rays which, being charged particles, have their motions continually altered by interactions with cosmic magnetic fields. Therefore, much can be learned about cosmic-ray sources and interactions by studying the spatial and energy distribution of the γ-rays they produce.

Recent laboratory studies of high-energy interactions utilizing proton accelerators have provided us with much information on the nature of these interactions up to $\sim$30-GeV energies. This information can be supplemented by studies of cosmic-ray interactions at higher energies occurring in the atmosphere. The knowledge gained from earthbound studies of high-energy interactions may then be applied to astronomical problems involving cosmic-ray and annihilation phenomena.

1-2 THE DETERMINATION OF GAMMA-RAY PRODUCTION SPECTRA DUE TO COSMIC-RAY INTERACTIONS WITH INTERSTELLAR AND INTERGALACTIC GAS

For our discussion of the production of cosmic γ-rays in high-energy interactions, it will be useful to define a source function $q(E_\gamma)$ such that

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$q(E_\gamma)dE_\gamma$ represents the number of γ-rays having energies in the range between E_γ and $E_\gamma + dE_\gamma$ produced per unit volume per unit time; $q(E_\gamma)$ can be written as the product

$$q(E_\gamma) = n\nu_c\langle\zeta\rangle \tag{1-1}$$

where n is the density of target nuclei in the medium considered, ν_c is the collision frequency per target nucleus, and $\langle\zeta\rangle$ is the average number (multiplicity) of γ-rays with energy E_γ produced per collision. If we define an average collision cross section $\langle S\rangle$, then ν_c is given by

$$\nu_c = \langle S\rangle 4\pi I_{\rm cr} \tag{1-2}$$

where $I_{\rm cr}$ is the average directional intensity of cosmic rays (in units of $\rm cm^{-2}\ s^{-1}\ sr^{-1}$), and the factor of 4π comes from integration over solid angle. Equations (1–1) and (1–2) then combine to give

$$q(E_\gamma) = 4\pi n\langle S\rangle\langle\zeta\rangle I_{\rm cr} \tag{1-3}$$

In general, for the production of γ-rays, ζ and S are functions of the primary cosmic-ray energy $E_{\rm pr}$. Therefore, we define

$$\sigma(E_\gamma|E_{\rm pr}) \equiv \zeta(E_{\rm pr})S(E_\gamma|E_{\rm pr}) \tag{1-4}$$

and write

$$q(E_\gamma) = 4\pi n \int dE_{\rm pr}\,\sigma(E_\gamma|E_{\rm pr})I(E_{\rm pr}) \tag{1-5}$$

Note that, in general, n and $I_{\rm cr}$ and therefore $q(E_\gamma)$ are functions of spatial location $\mathbf{r}$.

We can now derive an expression for the directional intensity of γ-rays of energy E_γ observed at the Earth. The flux reaching the Earth after being produced in a differential source volume dV located at a point $\mathbf{r}$ is given by

$$\begin{aligned} dF_\gamma(E_\gamma, \mathbf{r}) &= \frac{1}{4\pi r^2}\, q(E_\gamma, \mathbf{r})\ dV \\ &= \frac{1}{4\pi r^2}\, q(E_\gamma, \mathbf{r}) r^2\ d\Omega\ dr \end{aligned} \tag{1-6}$$

Thus, the directional intensity $dI_\gamma(E_\gamma, \mathbf{r})$ reduces to

$$dI_\gamma(E_\gamma, \mathbf{r}) = \frac{dF_\gamma(E_\gamma, \mathbf{r})}{d\Omega} = \frac{q(E_\gamma, \mathbf{r})\ dr}{4\pi} \tag{1-7}$$

and the total specific intensity is, therefore,

$$I_\gamma(E_\gamma) = \frac{1}{4\pi} \int dr\, q(E_\gamma, \mathbf{r}) \tag{1-8}$$

If, in addition, we take into account the effects of absorption of γ-rays by the medium, we must include an exponential absorption factor of the form $\exp\,[-\int dr\ \kappa(E_\gamma, \mathbf{r})]$, where $\kappa(E_\gamma, \mathbf{r})$ is the absorption of the medium per unit pathlength. Thus, the more general form of equation (1–8), which takes absorption into account, is

$$I_\gamma(E_\gamma) = \frac{1}{4\pi} \int dr\, q(E_\gamma, \mathbf{r})\ \exp\left[-\int_0^r dr'\, \kappa(E_\gamma, \mathbf{r}')\right] \tag{1-9}$$

We will now derive the source function $q(E_\gamma, \mathbf{r})$ for γ-ray production. By generalizing equation (1–5) to include collisions between various nuclides of cosmic rays and interstellar gas, we can express this function as

$$q(E_\gamma, \mathbf{r}) = 4\pi \sum_{j,k} n_j(\mathbf{r}) \int dE_k \sigma_{jk}(E_\gamma | E_k) I(E_k, \mathbf{r}) \tag{1-10}$$

so that the total γ-ray source function is obtained by summing over all the possible contributions from collisions between gas nuclei of type j and cosmic-ray nuclei of type k. We must also allow for various distinct γ-ray production interactions that can take place between two specified nuclei, j and k. To do this, we express the cross section $\sigma_{jk}(E_\gamma | E_k)$ as a sum over the contributions from various secondary particles which can be produced in the interaction and can subsequently decay to produce γ-rays. Denoting these particles as secondary particles of type s, we therefore write

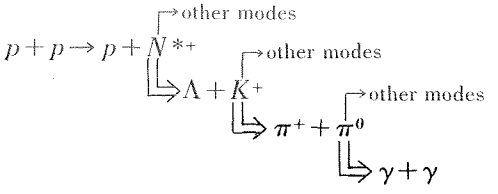
$$\sigma_{jk}(E_\gamma | E_k) = \sum_s \int dE_s \langle \zeta_{\gamma,s} \rangle f_s(E_\gamma | E_s) \sigma_{jk,s}(E_s | E_k) \tag{1-11}$$

where $\langle \zeta_{\gamma,s} \rangle$ is the average number of γ-rays produced by the decay of a secondary of type s and $f_s(E_\gamma | E_s)$ is an energy distribution function, normalized so that

$$\int_0^{E_s} dE_\gamma f_s(E_\gamma | E_s) = 1$$

which gives the probability that a γ-ray of energy E_γ will result from the decay of a secondary s having an energy E_s.

But this breakdown is still insufficient for the specification of γ-ray production interactions. We indicate this by citing the following example:



In this example we consider an interaction between two protons. We have chosen to follow a particular decay chain which results in the production of two γ-rays and is given by the double arrows. In this decay chain, we first produce an intermediate nonstrange isobar, N^{*+}, which decays in $\sim 10^{-23}$ s into a lambda hyperon and a positive K-meson (kaon). The kaon, in turn, decays in $\sim 10^{-8}$ s into a positive and a neutral π-meson (pion). The neutral pion then decays in $\sim 10^{-16}$ s, yielding two γ-rays. At each stage in the chain, there are branches into other possible decay modes which occur with specified probabilities (branching ratios). For example, the isobar could have decayed into a proton and a neutral pion and this alternate decay mode would have produced γ-rays having a different energy spectrum in the rest frame of the observer (hereafter referred to as the laboratory system (ls)).

Therefore, we must determine the sum over all possible γ-ray-producing decay modes d involving secondary particles of type s to evaluate the right-hand side of equation (1–11); i.e.,

$$\langle \zeta_{\gamma,\, s} \rangle f_s(E_\gamma \mid E_s) = \prod_d \langle \zeta_{\gamma,\, ds} \rangle f_{ds}(E_\gamma \mid E_s) \tag{1–12}$$

where, for decay modes involving l intermediate secondaries,

$$\langle \zeta_{\gamma,\, ds} \rangle = \zeta_{s'} R_{s'} \zeta_{s''} R_{s''} \cdots \zeta_{s(l)} R_{s(l)} \zeta_\gamma R_\gamma \tag{1–13}$$

Here, the quantities denoted by R indicate branching ratios for the production of various intermediate secondaries. The distribution function $f_{ds}(E_\gamma \mid E_s)$ is given by

$$f_{ds}(E_\gamma \mid E_s) = \int\int \cdots \int dE_{s'}\, dE_{s''} \cdots dE_{s(l)} f(E_\gamma \mid E_{s(l)}) f(E_{s(l)} \mid E_{s(l-1)}) \cdots f(E_{s'} \mid E_s) \tag{1–14}$$

Fortunately, long chain decay modes for γ-ray production turn out to have such small probabilities that they may be neglected.

1–3 GAMMA-RAY-PRODUCING DECAY MODES

We will consider the γ-rays produced from the following secondary particle decay channels: the electromagnetic decays

$$\pi^0 \rightarrow 2\gamma \tag{1–15}$$

$$\Sigma^0 \rightarrow \Lambda + \gamma \tag{1–16}$$

and also the weak decays

$$K^{\pm} \rightarrow \pi^{\pm} + \pi^0 \tag{1–17}$$

$$K_1^0 \rightarrow 2\pi^0 \tag{1–18}$$

$$K_2^0 \rightarrow 3\pi^0 \tag{1–19}$$

$$K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0 \tag{1–20}$$

$$\Lambda \rightarrow n + \pi^0 \tag{1–21}$$

and

$$\Sigma^+ \rightarrow p + \pi^0 \tag{1–22}$$

followed by $\pi^0 \rightarrow 2\gamma$. Table 1–1 shows some of the features of γ-ray production from the decays of p-p secondaries.

The hyperons can be created by associated production or in hyperon-antihyperon pairs; i.e., by the reactions

$$N + N \rightarrow N + Y + K + \cdots \tag{1–23}$$

and

$$N + N \rightarrow N + N + Y + \bar{Y} + \cdots \tag{1–24}$$

TABLE 1–1.—*Gamma-Ray Multiplicity and Branching Ratios From Hyperon and Meson Decays* [a]

Decay mode	Branching ratio R	Multiplicity ζ_γ	$R\zeta_\gamma$
$\pi^0 \rightarrow 2\gamma$	1.00	2	2.00
$\Sigma^0 \rightarrow \Lambda + \gamma$	1.00	1	1.00
$K^{\pm} \rightarrow \pi^{\pm} + \pi^0$	0.215	2	.430
$K_1^0 \rightarrow 2\pi^0$	$\frac{1}{2}$(0.311)	4	.622
$K_2^0 \rightarrow 3\pi^0$	$\frac{1}{2}$(0.265)	6	.795
$K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0$	$\frac{1}{2}$(0.114)	2	.114
$\Lambda \rightarrow n + \pi^0$	0.331	2	.662
$\Sigma^+ \rightarrow p + \pi^0$	0.510	2	1.02

[a] The reactions $\Xi^0 \rightarrow \Lambda + \pi^0$ and $\Omega^- \rightarrow \Xi^- + \pi^0$ may also occur, but production of the Ξ^0 and Ω^- hyperons is so rare that these decays will not be considered.

where the N denotes nucleon and the Y denotes hyperon. They may also arise from such reactions as

$$N+N\rightarrow N^*+N+\cdots \tag{1-25}$$

(where the asterisk denotes an isobar) followed by

$$N^*\rightarrow Y+K$$

and

$$N+N\rightarrow Y^*+N+K+\cdots \tag{1-26}$$

followed by

$$Y^*\rightarrow Y+\pi\text{'s}$$
$$\rightarrow N+\pi\text{'s}$$

which have end products similar to those of reactions (1–23) and (1–24), but whose resultant momentum and angular distributions are affected by the formation of the intermediate isobar. (Unless otherwise specified, the symbol N stands for nucleon, Y for any hyperon, K for any kaon, N^* for any nonstrange isobar, and Y^* for any strange isobar.) We may note also that at very high energies, as first pointed out by Fermi (1951), pions and kaon-antikaon pairs can be produced as what might be pictured as a quasithermal boson gas.

The experimental results of p–p interactions indicate the following features.

(1) For p–p machine data where the primary beam momentum is less than 10 GeV/c (what will be hereafter referred to as the low-energy region) secondaries are produced primarily through an intermediate stage involving isobar production; for example,[1]

$$p+p\rightarrow\Delta(1.238)+N$$

followed by (1–27)

$$\Delta(1.238)\rightarrow N+\pi$$

and

$$p+p\rightarrow\Sigma(1.385)+N+K$$

followed by (1–28)

$$\Sigma(1.385)\rightarrow\Lambda+\pi$$

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[1] We adopt here the contemporary notation for isobar designations. The quantity in the parentheses is the rest mass of the isobar in GeV/c^2. N^* and Δ represent nonstrange isobars having isospins 1/2 and 3/2, respectively. $\Lambda(\ )$ and $\Sigma(\ )$ represent isobars of strangeness -1 having isospins 0 and 1, respectively. $\Xi(\ )$ represents isobars of strangeness -2.

(2) The character of the interaction changes so that for "high" energies (primary proton energy greater than 10 GeV), the p–p interactions continue to yield only one or two isobars, with perhaps as many as four secondary mesons (K or π), but the bulk of the secondary mesons produced at high energies results from the formation of a quasithermal meson gas (Fermi, 1951; Heisenberg, 1952; and Landau, 1953), containing up to one-half the energy of the system and localized to one or two centers of excitation (sometimes referred to as fireballs), which move much more slowly in the center-of-mass system than do the baryon isobars. (See Daniel et al., 1963.)

1-4 KINEMATICS

The reactions that we consider involve energies at least of the order of magnitude of the rest masses of the particles involved. Therefore, we must use the results of the special theory of relativity to describe their kinematics. The following is a discussion of the kinematics that will be needed to describe the reactions listed in the previous section.

Special relativity makes use of a four-vector algebra with the corresponding four-momenta and four-velocities defined as the vectors

$$p^{(4)} = (\mathbf{p}, iE) \tag{1-29}$$

and

$$\beta^{(4)} = (\boldsymbol{\beta}, i) \tag{1-30}$$

In equations (1–29) and (1–30), the superscript denotes a four-vector, $\boldsymbol{\beta} \equiv \mathbf{v}/c$, and the speed of light c is taken equal to unity (as it will be hereafter).

The Lorentz transformation can be expressed as an orthogonal transformation that corresponds to a rotation through an angle ϕ in a complex four-vector space. Figure 1–1(a) shows that the β of the Lorentz trans-

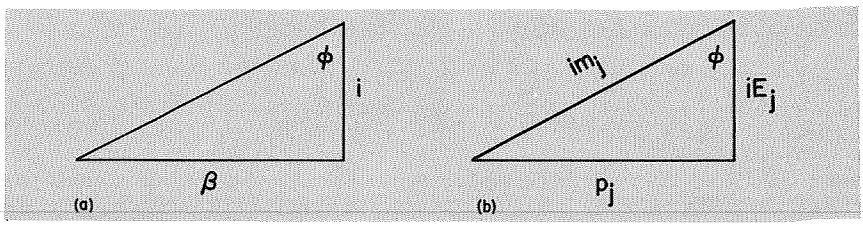

FIGURE 1–1.—Relativistic energy momentum and velocity triangles. (a) β triangle. (b) p–E triangle.

formation is then equal to tanh ϕ. It follows that, under this rotation, the "length" of the four-vector momentum

$$|p^{(4)}|^2 = |\mathbf{p}^2 - E^2| = m^2 \tag{1-31}$$

is an invariant and serves to define the proper mass m.

In figure 1–1(b) the subscript j serves as the index of each of the various particles involved in the reaction.

If we want to transform our kinetic equations from one system to another by performing a Lorentz transformation specified by the velocity β_2 (for example, we will later be concerned with transformations from center-of-momentum systems (cms) to laboratory systems (ls) and vice versa), the corresponding transformation, being simply a rotation in our complex four-space, can be performed merely by adding the angles $\phi = \tanh^{-1} \beta$. Thus, if we label quantities in the original system with the subscript 1, the transformation quantities with the subscript 2, and the transformed quantities with the subscript 3, we can write

$$\phi_3 = \phi_1 + \phi_2 \tag{1-32}$$

and

$$\beta_1 = \tanh \phi_1 \qquad \beta_2 = \tanh \phi_2 \tag{1-33}$$

therefore,

$$\beta_3 = \tanh (\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{1-34}$$

To find ϕ for the cms, we just add the four-momenta of all the particles involved in the reaction. Then, it follows from figure 1–1(b) that

$$\tanh \phi_{\text{cms}} = \beta_{\text{cms}} = \frac{\sum_j p_j}{\sum_j E_j} \tag{1-35}$$

From figure 1–1(a), we see that we can write

$$p = m \sinh \phi = \frac{m\beta}{(1-\beta^2)^{1/2}} \tag{1-36}$$

and

$$E = m \cosh \phi = \frac{m}{(1-\beta^2)^{1/2}} \tag{1-37}$$

We also introduce here a very useful quantity called the Lorentz factor γ, which is defined as

$$\gamma \equiv \frac{E}{m} = (1-\beta^2)^{-1/2} = \cosh \phi \tag{1-38}$$

As a particular case of equation (1–35), a case that we will be primarily concerned with later on, we consider a proton of energy E in the ls colliding with a proton at rest in the ls. We then find

$$\beta_{\text{cms}} = \frac{\sum p_j}{\sum E_j} = \frac{(E^2-m^2)^{1/2}}{E+m} = \left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} \tag{1-39}$$

and

$$\gamma_{\text{cms}} = (1-\beta_{\text{cms}}^2)^{-1/2} = [\tfrac{1}{2}(\gamma+1)]^{1/2} \tag{1-40}$$

which, for $\gamma \gg 1$, reduces to

$$\gamma_{\text{cms}} \simeq (\tfrac{1}{2}\gamma)^{1/2} \tag{1-41}$$

We can use the definition (1–38) with equation (1–32) to obtain directly the Lorentz transformation for γ. We find

$$\begin{aligned} \gamma_3 = \cosh \phi_3 &= \cosh (\phi_1 + \phi_2) \\ &= \cosh \phi_1 \cosh \phi_2 + \sinh \phi_1 \sinh \phi_2 \\ &= \gamma_1\gamma_2 + \gamma_1\beta_1 \gamma_2\beta_2 \\ &= \gamma_2[\gamma_1 + \beta_2(\gamma_1\beta_1)] \end{aligned} \tag{1-42}$$

which, together with the definition (1–38), yields the energy transformation relation

$$E_3 = \gamma_2[E_1 + \beta_2 p_1] \tag{1-43}$$

Similarly, we also obtain the momentum transformation relation

$$\begin{aligned} p_3 &= m \sinh \phi_3 \\ &= m \sinh \phi_1 \cosh \phi_2 + m \cosh \phi_1 \sinh \phi_2 \\ &= \gamma_2[p_1 + \beta_2(m\gamma_1)] \\ &= \gamma_2[p_1 + \beta_2 E_1] \end{aligned} \tag{1-44}$$

From the similarity of the β and p–E triangles of figure 1–1, we can recover the Lorentz contraction formulas by analogy with equations

(1–43) and (1–44). Therefore, we obtain

$$x_3 = \gamma_2(x_1 + \beta_2 t_1) \tag{1–45}$$

and

$$t_3 = \gamma_2(t_1 + \beta_2 x_1) \tag{1–46}$$

where x is understood to be the displacement in the direction of motion, just as p has been understood to be under a similar restraint. We will find it useful to generalize our transformations by looking at reactions in a two-dimensional x–y plane, x being taken to be at angle θ with the axis of the Lorentz transformation. Therefore, under the Lorentz transformation, $z_3 = z_1$. We can then define a direction of motion by the angular parameter θ, such that

$$\tan \theta_{1,3} = \frac{\dfrac{dy_{1,3}}{dt_{1,3}}}{\dfrac{dx_{1,3}}{dt_{1,3}}} \tag{1–47}$$

From the definition (1–47) and equations (1–45) and (1–46), we immediately obtain the transformation relation for θ:

$$\tan \theta_3 = \frac{\sin \theta_1}{\gamma_2[\cos \theta_1 + (\beta_2/\beta_1)]} \tag{1–48}$$

Equations (1–43) through (1–45) can now be further generalized by the replacements

$$p \rightarrow p \cos \theta \tag{1–49}$$

and

$$x \rightarrow x \cos \theta \tag{1–50}$$

which define the axis of the Lorentz transformation, where p and x now represent the total magnitudes of these vectors in three-space.

We now consider the kinematics of two-body collisions and the kinematics of particle decay. For our purposes, we may limit ourselves to two- and three-body decays, since we need discuss only the decay reactions (1–15) through (1–22) and isobar decays where not more than three particles are produced (Morrison, 1963). It should be noted that of the decay modes (1–15) through (1–22), only the $K^{\pm}$ and K_2^0 channels involve three decay products. In those cases, the three particles produced have essentially identical masses, which simplifies the treatment.

First, we consider the reaction

$$a + b \rightarrow c + d + \cdots \tag{1–51}$$

where particle b is the target particle, at rest in the ls. Since the length of the total four-momentum is invariant, we have

$$[\Sigma p_{\text{ls}}^{(4)}]^2 = [\Sigma p_{\text{cms}}^{(4)}]^2 \tag{1-52}$$

or

$$p_a^2 - (E_a + m_b)^2 = -(E_a' + E_b')^2 \tag{1-53}$$

where quantities in the cms will be denoted by primes. We now let E' represent the total energy in the cms. Then, from equation (1–53)

$$\begin{aligned} E' = E_a' + E_b' &= [(E_a + m_b)^2 - p_a^2]^{1/2} \\ &= [m_a^2 + m_b^2 + 2m_b E_a]^{1/2} \end{aligned} \tag{1-54}$$

Using the four-momentum invariant inner product

$$[p_a^{(4)} \cdot (p_a^{(4)} + p_b^{(4)})]_{\text{ls}} = [p_a^{(4)} \cdot (p_a^{(4)} + p_b^{(4)})]_{\text{cms}} \tag{1-55}$$

we obtain expressions for the individual cms energies of particles a and b. Thus, we obtain

$$E_a' = \frac{E'^2 + m_a^2 - m_b^2}{2E'} \tag{1-56}$$

and

$$E_b' = \frac{E'^2 + m_b^2 - m_a^2}{2E'} \tag{1-57}$$

To obtain the threshold energies for the creation of particles c, d, . . ., we write

$$E_{\text{th}}' = m_a + m_b + \Delta m = m_c + m_d + \cdots \tag{1-58}$$

where Δm is the mass difference between the incoming and outgoing particles as defined by equation (1–58). In the case that we have been considering, where one particle is at rest in the ls, we note that the threshold energy is the energy of the incoming particle which is equivalent to just enough available energy in the cms (given by eq. (1–54)) to produce the rest mass of all the outgoing particles (given by eq. (1–58)). Thus, we equate equations (1–54) and (1–58) to obtain

$$m_a + m_b + \Delta m = [m_a^2 + m_b^2 + 2m_b (E_a)_{\text{th}}]^{1/2} \tag{1-59}$$

which reduces to

$$(E_a)_{\text{th}} = \left(1 + \frac{m_a}{m_b} + \frac{\Delta m}{2m_b}\right)\Delta m + m_a \tag{1-60}$$

In terms of kinetic energy $T = (E - m) = (\gamma - 1)m$, we have

$$T_{\text{th}} = \left(1 + \frac{m_a}{m_b} + \frac{\Delta m}{2m_b}\right)\Delta m \tag{1-61}$$

Table 1–2 lists cms energies for a p–p system as a function of laboratory energy, derived from equation (1–40).

Let us now consider the decay of a particle of mass M into two particles of mass m_a and m_b, respectively. From equations (1–56) and (1–57), we find that in the rest system of mass M

$$E'_a = \frac{M^2 + m_a^2 - m_b^2}{2M} \tag{1-62}$$

TABLE 1–2.—*Transformation From Laboratory Energy to cms Energy in* p–p *Collisions*

$T_{\text{ls}} = E_p - m_p$, GeV	$\gamma_p = E_p/m_p$	γ_{cms}	$Q = 2m_p(\gamma_c - 1)$, GeV
0.0	1.00	1.00	0.0
0.5	1.53	1.13	.235
1.0	2.07	1.24	.447
1.5	2.60	1.34	.641
2.0	3.13	1.44	.821
2.5	3.67	1.53	.989
3.0	4.20	1.61	1.149
3.5	4.73	1.69	1.300
4.0	5.26	1.77	1.444
4.5	5.80	1.84	1.583
5.0	6.33	1.91	1.716
5.5	6.86	1.98	1.844
6.0	7.40	2.05	1.968
6.5	7.93	2.11	2.088
7.0	8.46	2.18	2.205
7.5	9.00	2.23	2.318
8.0	9.53	2.29	2.428
8.5	10.06	2.35	2.536
9.0	10.59	2.41	2.641
9.5	11.13	2.46	2.744
10.0	11.66	2.52	2.844

and

$$E'_b = \frac{M^2 + m_b^2 - m_a^2}{2M} \tag{1-63}$$

The rest system of mass M must also be the cms of the two decay particles. Therefore, they are given equal and opposite momenta of magnitude p' in this system. Using equations (1–43) and (1–49), we obtain the Lorentz transformation to an ls where the original mass M had an energy $E_M = \gamma M$. This transformation is given by

$$E_{a,b} = \gamma(E'_{a,b} + \beta p' \cos\theta') \tag{1-64}$$

Equations (1–62) through (1–64) then yield the energies of the decay particles in the ls. These energies are given by

$$E_{a,b} = \frac{E_M}{M}\left[\left(\frac{M^2 + m_{a,b}^2 - m_{b,a}^2}{2M}\right) + \left(1 - \frac{M^2}{E_M^2}\right)^{1/2} \times p'(M, m_a, m_b)\cos\theta'\right] \tag{1-65}$$

where $p'(M, m_a, m_b)$ is given implicitly by the conservation of energy relation

$$(p'^2 + m_a^2)^{1/2} + (p'^2 + m_b^2)^{1/2} = M \tag{1-66}$$

Equation (1–65) immediately yields the following results:

(1) If $\langle\cos\theta'\rangle = 0$, i.e., if we have equal emission in the forward and backward directions in the cms,

$$\langle E_{a,b}\rangle = E_M\left(\frac{M^2 + m_{a,b}^2 - m_{b,a}^2}{2M^2}\right) \tag{1-67}$$

(2) If particle M decays isotropically, i.e., if there is equal probability of emission in all directions in the cms, the normalized angular distribution function for the emitted particles is given by

$$f(\theta')\, d\theta' = \frac{d\Omega(\theta')}{4\pi} = \frac{1}{2}\sin\theta'\, d\theta' \tag{1-68}$$

We will now make some useful substitutions in equation (1–65) to write it in the form

$$E(\theta') = E_M(\eta + \beta\kappa\cos\theta') \tag{1-69}$$

so that

$$dE(\theta') = \beta E_M \kappa \sin \theta' \, d\theta' \tag{1-70}$$

where

$$\eta \equiv \mu/M$$

is the dimensionless ratio of the cms energy of the daughter particle to the mass of the parent particle, and $\kappa \equiv p'/M$ is the dimensionless ratio of the cms momentum of the daughter particle to the mass of the parent particle. Consequently,

$$f(E)\, dE = \frac{f(\theta')}{dE/d\theta'}\, dE = \frac{1}{2\beta\kappa E_M}\, dE \tag{1-71}$$

As a check on the normalization of $f(E)$, we note that

$$\int_{E_{\min}}^{E_{\max}} f(E)\, dE = \frac{1}{2\beta\kappa E_M} \int_{E_M(\eta-\beta\kappa)}^{E_M(\eta+\beta\kappa)} dE = \frac{2\beta\kappa E_M}{2\beta\kappa E_M} = 1 \tag{1-72}$$

This function is graphed in figure 1–2.

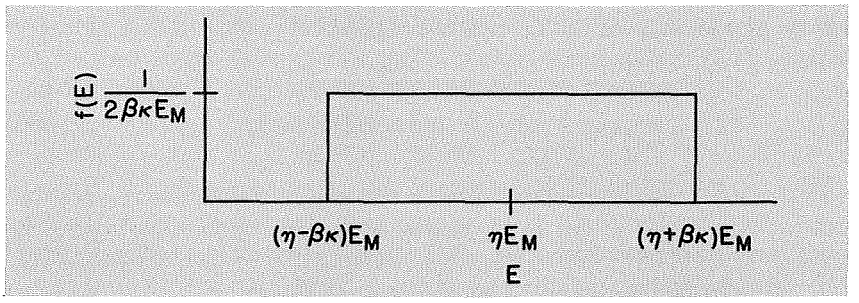

FIGURE 1–2.—The form of the energy distribution function for a particle formed isotropically from two-body decay.

(3) If we have a case where a particle of mass M decays into two particles of equal mass m, e.g., $K_1^0 \rightarrow 2\pi^0$, equation (1–65) reduces to

$$E_m = \frac{1}{2} E_M + \left(\frac{E_M^2}{M^2} - 1\right)^{1/2} \left(\frac{1}{4} M^2 - m^2\right)^{1/2} \cos \theta' \tag{1-73}$$

If, in addition, the decay is isotropic in θ', the average energy $\langle E_m \rangle$ and the half-width of the energy distribution Δ_m of decay particles having mass m are given by

$$\langle E_m \rangle = \frac{1}{2} E_M \tag{1-74}$$

and

$$\Delta_m = \left[\left(\frac{E_M^2}{M^2} - 1 \right) \left(\frac{1}{4} M^2 - m^2 \right) \right]^{1/2} \tag{1-75}$$

We find a discrete energy for the decay particles in the ls; i.e., $\Delta_m = 0$, only for two cases: (*a*) $E_M^2/M^2 = 1$, i.e., the particle of mass M is at rest in the ls; and (*b*) $m = \frac{1}{2}M$, i.e., the decay particles have no kinetic energy in the cms.

(4) If particle M decays into two particles, one of which has mass zero, e.g., $\Sigma^0 \rightarrow \Lambda + \gamma$, equation (1–65) reduces to

$$E_m = \frac{E_M}{M} \left[\frac{M^2 + m^2}{2M} + \left(1 - \frac{M^2}{E_M^2} \right)^{1/2} p'(M, m) \cos \theta' \right] \tag{1-76}$$

and

$$E_0 = \frac{E_M}{M} \left[\frac{M^2 - m^2}{2M} + \left(1 - \frac{M^2}{E_M^2} \right)^{1/2} p'(M, m) \cos \theta' \right] \tag{1-77}$$

Since, for a particle of mass zero $p = E$, we find

$$p'(M, m) = \frac{M^2 - m^2}{2M} \tag{1-78}$$

and thus

$$E_0 = \frac{E_M}{M} \left(\frac{M^2 - m^2}{2M} \right) \left[1 + \left(1 - \frac{M^2}{E_M^2} \right)^{1/2} \cos \theta' \right]$$

$$= E_M \eta(M, m) (1 + \beta_M \cos \theta') \tag{1-79}$$

(5) If both decay particles have mass zero, e.g., $\pi^0 \rightarrow 2\gamma$, equation (1–65) reduces to

$$E_0 = \tfrac{1}{2} E_M \left[1 + \left(1 - \frac{M^2}{E_M^2} \right)^{1/2} \cos \theta' \right]$$

$$= \tfrac{1}{2} M \gamma_M (1 + \beta_M \cos \theta') \tag{1-80}$$

In particular, for $M^2/E^2 \ll 1$, i.e., for ultrarelativistic particles, equation (1–80) reduces to

$$E_0 = E_M \cos^2 (\tfrac{1}{2}\theta') \tag{1-81}$$

and the energy distribution then has the corresponding parameters

$$\langle E_0 \rangle = \tfrac{1}{2} E_M \qquad \Delta_0 = \tfrac{1}{2} E_M \tag{1-82}$$

This function is shown in figure 1–3.

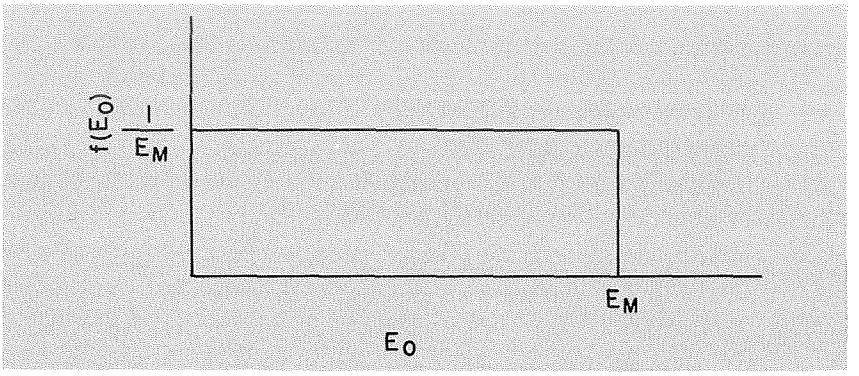

FIGURE 1–3.—The form of the energy distribution function for γ-rays resulting from the decay of relativistic neutral pions.

The ultrarelativistic form for equation (1–79) yields a distribution function that is similar in form to that of figure 1–3, with the resultant E_0 given by the simplified expression

$$E_0 = 2E_M\eta(M, m)\ \cos^2(\tfrac{1}{2}\theta') \tag{1–83}$$

with $\eta(M, m)$ defined by equation (1–79).

It should be noted that the kinematics of equation (1–65) can be applied to isobar decay as well as to the weak and electromagnetic decays of the various particles. Thus, we can make a distinction between the three-body final state in a reaction such as

$$p + p \rightarrow p + \Lambda + K^+ \tag{1–84}$$

and the double two-body kinematics problem posed by a reaction such as

$$p + p \rightarrow p + N^{*+} \tag{1–85}$$

followed by

$$N^{*+} \rightarrow \Lambda + K^+ \tag{1–86}$$

Clearly, the kinematics of reaction (1–86) is more restrained than is that of reaction (1–85), even though the mass of N^* has an associated uncertainty $\hbar/\tau_d$, where τ_d is the decay time of the isobar and $\hbar$ is Planck's constant divided by 2π.

1-5 THE EVALUATION OF THE SOURCE FUNCTIONS DUE TO VARIOUS DECAY MODES

1-5a The Decay Mode $\pi^0 \rightarrow 2\gamma$

Equation (1–80) can be used to write an expression for the energy of a γ-ray produced by the decay $\pi^0 \rightarrow 2\gamma$. This energy is then given by

$$E_\gamma = \tfrac{1}{2} m_\pi \gamma_\pi (1 + \beta_\pi \cos \theta') \tag{1–87}$$

Equation (1–71) reduces to give the γ-ray energy-distribution function. This result, together with the limits implied by equation (1–87), yields the Green's function for the decay $f_{ds}(E_\gamma | E_s)$ of equation (1–14). The result is

$$f(E_\gamma | E_\pi) = \begin{cases} (E_\pi^2 - m_\pi^2)^{-1/2} & \text{for } \tfrac{1}{2} E_\pi (1 - \beta_\pi) \leq E_\gamma \leq \tfrac{1}{2} E_\pi (1 + \beta_\pi) \\ 0 & \text{otherwise} \end{cases} \tag{1–88}$$

We note also that, for the neutral-pion decay,

$$R_{\gamma,\, \pi^0 \rightarrow 2\gamma} \simeq 1 \tag{1–89}$$

and

$$\zeta_{\gamma,\, \pi^0 \rightarrow 2\gamma} = 2 \tag{1–90}$$

Thus, the contribution of neutral-pion decay to the source function of equation (1–10) reduces to

$$q(E_\gamma, \mathbf{r}) = 4\pi n(\mathbf{r}) \int dE_p \, I(E_p, \mathbf{r}) \int_{E_{\pi,\,\min}}^{E_{\pi,\,\max}} dE_\pi \, \sigma(E_\pi | E_p) \cdot 2 \cdot 1 \cdot (E_\pi^2 - m_\pi^2)^{-1/2} \tag{1–91}$$

where the limits $E_{\pi,\max}$ and $E_{\pi,\min}$ can be found from the following considerations.

The quantity $E_{\pi,\max}$ is the maximum energy that a pion can have and still produce a γ-ray of energy E_γ. We note that as $E \rightarrow \infty$ the ultra-relativistic form of the pion/γ-ray energy relation given by equation (1–81) becomes valid, so that

$$E_{\pi,\,\max} = \frac{E_\gamma}{[\cos^2 (\tfrac{1}{2}\theta')]_{\min}} \rightarrow \infty \tag{1–92}$$

since

$$[\cos^2 (\tfrac{1}{2}\theta')]_{\min} = 0$$

From conservation of energy, we note that, for the two γ-rays produced in the decay,

$$E_\pi = E_{\gamma,a} + E_{\gamma,b} \tag{1-93}$$

In the extreme case where the γ-rays are emitted in the direction of motion, $\cos\theta' = 1, -1$,

$$E_{\gamma,\min} = \tfrac{1}{2}E_\pi(1-\beta_\pi) \tag{1-94}$$

and

$$E_{\gamma,\max} = \tfrac{1}{2}E_\pi(1+\beta_\pi) \tag{1-95}$$

Thus, corresponding to the highest energy $E_{\gamma,\max}$ that can be emitted by a pion of energy E_π, a γ-ray of energy $E_{\gamma,\min}$ is also emitted such that

$$E_{\gamma,\max} + E_{\gamma,\min} = \tfrac{1}{2}E_\pi(1+\beta_\pi) + \tfrac{1}{2}E_\pi(1-\beta_\pi) = E_\pi \tag{1-96}$$

We can use equations (1–94) and (1–95) to express $E_{\gamma,\min}$ in terms of $E_{\gamma,\max}$ by noting that

$$(E_{\gamma,\min})(E_{\gamma,\max}) = \tfrac{1}{4}E_\pi^2(1-\beta_\pi^2) = \tfrac{1}{4}m_\pi^2 \tag{1-97}$$

so that

$$E_\pi = E_{\gamma,\max} + E_{\gamma,\min} = E_{\gamma,\max} + \frac{m_\pi^2}{4E_{\gamma,\max}} \tag{1-98}$$

This criterion can then be reversed to put a lower limit on the pion energy integration by writing equation (1–98) as

$$E_{\pi,\min} = E_\gamma + \frac{m_\pi^2}{4E_\gamma} \tag{1-99}$$

where $E_{\pi,\min}$ is now the minimum energy that a pion needs to produce a γ-ray of energy E_γ. Thus, equation (1–91) reduces to

$$q_{\pi^0}(E_\gamma, \mathbf{r}) = 8\pi n(\mathbf{r}) \int dE_p I(E_p, \mathbf{r}) \int_{E_\gamma + m_\pi^2/4E_\gamma}^{\infty} dE_\pi \frac{\sigma(E_\pi|E_p)}{(E_\pi^2 - m_\pi^2)^{1/2}} \tag{1-100}$$

1–5b The Decay Mode $\Sigma^0 \rightarrow \Lambda + \gamma$

The energy distribution function and energy transformation equation for the Σ^0 decay are given by equations (1–71) and (1–79) as follows:

$$E_\gamma = E_{\Sigma^0}\,\eta(\Sigma^0, \Lambda)(1+\beta_\Sigma \cos\theta') \tag{1-101}$$

and

$$f(E_\gamma|E_{\Sigma^0})\,dE_\gamma = \frac{dE_\gamma}{2E_{\Sigma^0}\beta_{\Sigma^0}\eta(\Sigma^0,\,\Lambda)} \tag{1–102}$$

where

$$\eta(\Sigma^0,\,\Lambda) = \frac{m_{\Sigma^0}^2 - m_\Lambda^2}{2m_{\Sigma^0}^2} \tag{1–103}$$

The evaluation of the source function for the Σ^0 decay can then be made in a manner similar to that used for the π^0 decay. For the Σ^0 decay, we have

$$R_{\gamma,\,\Sigma^0\to\Lambda+\gamma} \simeq 1 \tag{1–104}$$

and

$$\zeta_{\gamma,\,\Sigma^0\to\Lambda+\gamma} = 1 \tag{1–105}$$

The distribution function $f(E_\gamma|E_{\Sigma^0})$ is then written explicitly in terms of E_Σ as

$$f(E_\gamma|E_{\Sigma^0}) = \begin{cases} \dfrac{1}{2\eta(\Sigma^0,\,\Lambda)\,(E_\Sigma^2 - m_\Sigma^2)^{1/2}} & \\ & \text{for } E_{\Sigma^0}\eta(1-\beta_{\Sigma^0}) \leqslant E_\gamma \leqslant E_{\Sigma_0}\eta(1-\beta_{\Sigma^0}) \\ 0 & \text{otherwise} \end{cases} \tag{1–106}$$

Thus, equation (1–10) reduces to

$$q_{\Sigma^0}(E_\gamma,\,\mathbf{r}) = \frac{2\pi}{\eta(\Sigma^0,\,\Lambda)}\,n(\mathbf{r}) \int dE_p\, I(E_p,\,\mathbf{r}) \int_{E_{\Sigma,\,\min}}^{E_{\Sigma,\,\max}} dE_\Sigma\, \frac{\sigma(E_\Sigma|E_p)}{(E_\Sigma^2 - m_\Sigma^2)^{1/2}} \tag{1–107}$$

where the limits $E_{\Sigma,\,\max}$ and $E_{\Sigma,\,\min}$ are deduced from equation (1–101). By analogy with equations (1–97) and (1–98), we find for the case of Σ^0 decay

$$(E_{\gamma,\,\min})\,(E_{\gamma,\,\max}) = E_\Sigma^2\eta^2(1-\beta_{\Sigma^0}^2) = \eta^2 m_\Sigma^2 = \mu^2 \tag{1–108}$$

where μ, as before, is the cms energy of the γ-ray. From equation (1–101), we find

$$E_{\Sigma,\,\max} = \frac{E_\gamma}{\eta(1-\beta_\Sigma)} \to \infty \qquad \text{as } \beta_\Sigma \to 1 \tag{1–109}$$

and

$$E_{\Sigma,\,\min} = \frac{E_\gamma}{\eta(1+\beta_\Sigma)} \to \frac{E_\gamma}{2\eta} \qquad \text{as } \beta_\Sigma \to 1 \tag{1–110}$$

We will later show that the case $\beta_\Sigma \to 1$ is the only one of importance to cosmic γ-ray production from Σ^0 decay. Thus, the source function for the production of cosmic γ-rays through the decay of secondary Σ^0 particles produced in high-energy cosmic p–p collisions is given by

$$q_{\Sigma^0}(E_\gamma, \mathbf{r}) = \frac{2\pi}{\eta(\Sigma^0, \Lambda)} n(\mathbf{r}) \int dE_p \, I(E_p, \mathbf{r}) \int_{\frac{E_\gamma}{2\overline{\eta}}}^{\infty} dE_\Sigma \frac{\sigma(E_\Sigma|E_p)}{(E_\Sigma^2 - m_\Sigma^2)^{1/2}} \tag{1-111}$$

where $M_\Sigma = 1.192$ GeV and $\mu = 7.466 \times 10^{-2}$ GeV, so $\eta^{-1} = 15.96$.

1–5c The Decay Mode $K_1^0 \to 2\pi^0 \to 4\gamma$

For this two-step process, we must express the Green's function, $f_{K_1^0 \to 4\gamma}(E_\gamma|E_K)$ in the form

$$f_{K_1^0 \to 4\gamma}(E_\gamma|E_K) = \int dE_{\pi^0} f_{\pi^0 \to 2\gamma}(E_\gamma|E_{\pi^0}) \, f_{K_1^0 \to 2\pi^0}(E_{\pi^0}|E_K) \tag{1-112}$$

where, as we have already shown (eq. (1–88)),

$$f_{\pi^0 \to 2\gamma}(E_\gamma|E_{\pi^0}) = \frac{1}{(E_\pi^2 - m_\pi^2)^{1/2}} \qquad \text{for } \tfrac{1}{2}E_\pi(1-\beta_\pi) \leq E_\gamma \leq \tfrac{1}{2}E_\pi(1+\beta_\pi)$$

$$= 0 \qquad \text{otherwise} \tag{1-113}$$

The function $f_{K_1^0 \to 2\pi^0}(E_\pi|E_K)$ can be derived from equations (1–71) and (1–73). Since the decay is isotropic in the cms, we have

$$f_{K_1^0 \to 2\pi^0}(E_\pi|E_K) = \tfrac{1}{2}\Delta_K^{-1} \qquad \text{for } \tfrac{1}{2}E_K - \Delta_K \leq E_\pi \leq \tfrac{1}{2}E_K + \Delta_K$$

$$= 0 \qquad \text{otherwise} \tag{1-114}$$

where Δ_K is the half-width of the square distribution function. From equation (1–75), we have

$$\Delta_K = \left[\left(\frac{E_K^2}{m_K^2} - 1\right)\left(\frac{m_K^2}{4} - m_\pi^2\right)\right]^{1/2} \tag{1-115}$$

Figure 1–4 schematically illustrates the relationships between E_{K^0}, E_{π^0}, and E_γ for the decay mode $K_1^0 \to 2\pi^0 \to 4\gamma$.

Referring to figure 1–4, we see that a neutral kaon of energy E_K (point A on the E_{K^0} scale) will decay to produce two neutral pions having

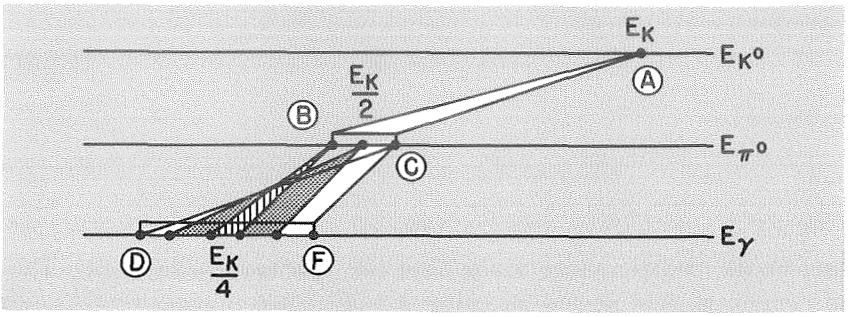

FIGURE 1–4. — The relationships between E_{K^0}, E_{π^0}, and E_γ for the decay mode $K_1^0 \rightarrow 2\pi^0 \rightarrow 4\gamma$.

energies in the range between B and C (when it decays via the K_1^0 mode), where

$$E_\pi(B) = \tfrac{1}{2}E_K - \Delta_K \tag{1–116}$$

and

$$E_\pi(C) = \tfrac{1}{2}E_K + \Delta_K \tag{1–117}$$

The resultant γ-rays have energies on the E_γ scale that lie between points D and F, such that

$$E_\gamma(D) = \frac{1}{2}\left(\frac{E_K}{2} + \Delta_K\right) - \Delta_\pi(C) \tag{1–118}$$

and

$$E_\gamma(F) = \frac{1}{2}\left(\frac{E_K}{2} + \Delta_K\right) + \Delta_\pi(C) \tag{1–119}$$

We have already found from equation (1–99) that

$$E_{\pi,\,\min} = E_\gamma + \frac{m_\pi^2}{4E_\gamma}$$

Now we must determine the minimum energy that the K_1^0 meson state needs to produce a neutral pion of energy E_π. If we define[2]

$$\kappa \equiv \left(1 - \frac{4m_\pi^2}{m_K^2}\right)^{1/2} \simeq 0.840 \tag{1–120}$$

[2] Note important difference with equation (1–69). From equation (1–120) on, $\kappa \equiv 2p'/M$.

equation (1–115) yields

$$\Delta_K = \tfrac{1}{2} E_K \beta \kappa \tag{1–121}$$

followed by

$$(E_{\pi,\,\max})(E_{\pi,\,\min}) = \tfrac{1}{4} E_K^2 (1 - \kappa^2 \beta_K^2) \tag{1–122}$$

We can see that equation (1–122) is analogous to equation (1–97) and would further reduce to the limit $\frac{1}{4} m_K^2$ if κ approached unity. This, of course, is as it should be, since a value for κ of unity would imply a decay into massless particles. Thus, equation (1–122) is a generalization of equation (1–97) for the case of decay into two particles having finite and equal masses. We also note from equation (1–121) that a value of zero for κ implies a monoenergetic spectrum of neutral pions, in agreement with our previous discussion of equation (1–75).

Using equations (1–42) and (1–49), we find

$$E_K = \left(\frac{m_K^2}{2m_\pi^2}\right)(1 + \beta_\pi \kappa \cos \theta') E_\pi \tag{1–123}$$

and therefore kaons capable of producing pions of energy E_π are restricted to the energy range

$$\left(\frac{m_K^2}{2m_\pi^2}\right)(1 - \beta_\pi \kappa) E_\pi \leq E_K \leq \left(\frac{m_K^2}{2m_\pi^2}\right)(1 + \beta_\pi \kappa) E_\pi \tag{1–124}$$

In the nonrelativistic and extreme relativistic cases, where $\beta_K \ll 1$ and $\beta_K \simeq 1$, we can use equations (1–115) and (1–116) to determine the limits on the range of kaons of energy E_K which contribute pions of energy E_π as

$$E_K \simeq 2E_\pi \qquad \text{for } \beta_K \ll 1 \tag{1–125}$$

and

$$E_{K,\,\max} \simeq 12.8\, E_\pi \tag{1–126}$$

$$E_{K,\,\min} \simeq 1.09\, E_\pi \tag{1–127}$$

for $\beta_K \simeq \beta_\pi \simeq 1$.

Finally, for the decay mode $K_1^0 \rightarrow 4\gamma$ we have

$$R_{K_1^0 \rightarrow 4\gamma} \simeq 0.156 \tag{1–128}$$

and

$$\zeta_{K_1^0 \rightarrow 4\gamma} = 4 \tag{1–129}$$

We can thus reduce equation (1–10) in this case to obtain the source function

$$q_{K_1^0}(E_\gamma, \mathbf{r}) = 3.11\pi n(\mathbf{r}) \int dE_p\, I(E_p, \mathbf{r}) \int_{E_\gamma + (m_\pi^2/4E_\gamma)}^{\infty} \frac{dE_\pi}{(E_\pi^2 - m_\pi^2)^{1/2}}$$

$$\times \int_{(6.89-5.78\beta_\pi)E_\pi}^{(6.89+5.78\beta_\pi)E_\pi} \frac{dE_K \sigma(E_{K^0}|E_p)}{(E_{K^0}^2 - m_{K^0}^2)^{1/2}} \quad (1\text{–}130)$$

since

$$\sigma(E_{K_1^0}|E_p) = \tfrac{1}{2}\sigma(E_{K^0}|E_p) \quad (1\text{–}131)$$

1–5d The Decay Mode $K^\pm \to \pi^\pm + \pi^0 \to \pi^\pm + 2\gamma$

The kinematics for this decay are exactly the same as for the decay $K_1^0 \to 2\pi^0 \to 4\gamma$. If we neglect the small difference between the kinetic energies released in the above decays ($Q_{K^0} = 228.0$ MeV, $Q_{K^\pm} = 219.2$ MeV), the only significant difference for our purposes is that the $K^\pm$ two-body decay produces only half as many γ-rays as the $K_1^0 \to 4\gamma$ decay. The relevant numerical quantities are

$$\kappa \simeq 0.84 \quad (1\text{–}132)$$

$$R \simeq 0.215 \quad (1\text{–}133)$$

$$\zeta = 2 \quad (1\text{–}134)$$

for

$$K^\pm \to \pi^\pm + \pi^0$$

(See eqs. (1–120), (1–128), and (1–129).) The expression for the source functions from $K^\pm$ decay is then similar in form to equation (1–130) and is given by

$$q_{K^\pm}(E_\gamma, \mathbf{r}) = 2.15\pi n(\mathbf{r}) \int dE_p\, I(E_p, \mathbf{r})$$

$$\times \int_{E_\gamma + (m_\pi^2/4E_\gamma)}^{\infty} \frac{dE_p}{(E_\pi^2 - m_\pi^2)^{1/2}} \int_{(6.78-5.69\beta_\pi)E_\pi}^{(6.78+5.69\beta_\pi)E_\pi} \frac{dE_K \sigma(E_{K^\pm}|E_p)}{(E_{K^\pm}^2 - m_{K^\pm}^2)^{1/2}} \quad (1\text{–}135)$$

1–5e The Decay Mode $\Lambda \to n + \pi^0$ for Unpolarized Λ Particles

The energy-distribution function and energy-transformation equation for unpolarized $\Lambda \to n + \pi^0$ decay are given by equations (1–65) and (1–71) as follows:

$$f_{\Lambda \to n+\pi^0}(E_\pi|E_\Lambda) = \frac{1}{2\Delta_K} \qquad \text{for } E_\Lambda \eta(\Lambda, \pi^0) - \Delta_{\Lambda\pi^0} \leq E_\pi \leq E_\Lambda \eta(\Lambda, \pi^0) + \Delta_{\Lambda\pi^0}$$

$$0 \qquad \text{otherwise} \quad (1\text{–}136)$$

and

$$E_\pi = E_\Lambda \eta(\Lambda, \pi^0) + \Delta_{\Lambda\pi^0} \cos\theta' \tag{1-137}$$

where

$$\eta(\Lambda, \pi^0) = \frac{m_\Lambda^2 + m_\pi^2 - m_n^2}{2m_\Lambda^2} \simeq 0.152 \tag{1-138}$$

$$\Delta_{\Lambda\pi^0} = \tfrac{1}{2} E_\Lambda \beta \kappa_{\Lambda,\pi^0} \tag{1-139}$$

and

$$\kappa_{\Lambda\pi^0} = \frac{2p'}{m_\Lambda} \tag{1-140}$$

and p' being given implicitly by

$$(m_\pi^2 + p'^2)^{1/2} + (m_n^2 + p'^2)^{1/2} = m_\Lambda \tag{1-141}$$

Noting that

$$\left.\begin{aligned} m_\Lambda &\simeq 1115.44 \text{ MeV} \\ m_n &\simeq 939.55 \text{ MeV} \end{aligned}\right\} \tag{1-142}$$

and

$$m_{\pi^0} \simeq 134.97 \text{ MeV}$$

we find from equation (1–141) that

$$p' \simeq 103.6 \qquad \text{MeV}/c \tag{1-143}$$

and

$$\kappa_{\Lambda\pi} \simeq 0.186$$

Thus, we obtain

$$f_{\Lambda\to n+\pi^0}(E_\pi | E_\Lambda) = \frac{5.38}{(E_\Lambda^2 - m_\Lambda^2)^{1/2}} \tag{1-144}$$

Using the same techniques used to derive equation (1–124), we can now find the limits on the E_Λ distribution.

From equations (1–42) and (1–49), we obtain

$$\gamma_\Lambda = \gamma_\pi \gamma'_\pi (1 + \beta_\pi \beta'_\pi \cos\theta') \tag{1-145}$$

or

$$E_\Lambda = E_\pi E'_\pi \left(\frac{m_\Lambda}{m_\pi^2}\right)^2 (1 + \beta_\pi \beta'_\pi \cos\theta') = \left(\frac{m_\Lambda}{m_\pi^2}\right)(E'_\pi + \beta_\pi p'_\pi \cos\theta') E_\pi \tag{1-146}$$

By using equations (1–138) and (1–140), we may further reduce equation (1–146) to the form

$$E_\Lambda = \left(\frac{m_\Lambda}{m_\pi}\right)^2 \left[\eta(\Lambda, \pi^0) + \beta_\pi \frac{\kappa(\Lambda, \pi^0)}{2} \cos\theta'\right] E_\pi \qquad (1\text{–}147)$$

so that

$$\left(\frac{m_\Lambda}{m_\pi}\right)^2 \left(\eta - \beta_\pi \frac{\kappa}{2}\right) E_\pi \leq E_\Lambda \leq \left(\frac{m_\Lambda}{m_\pi}\right)^2 \left(\eta + \beta_\pi \frac{\kappa}{2}\right) E_\pi \qquad (1\text{–}148)$$

Using the numerical values given in equations (1–138) and (1–143), we then find

$$(10.52 - 6.44\beta_\pi) E_\pi \leq E_\Lambda \leq (10.52 + 6.44\beta_\pi) E_\pi \qquad (1\text{–}149)$$

Equation (1–147) is analogous to equation (1–123) and reduces to equation (1–123) for the case of decay into two particles of equal mass, where $\eta = 1/2$. The nonrelativistic and extreme relativistic limits of equation (1–137) for the parent Λ yield

$$E_\Lambda = 6.58 E_\pi \qquad \text{for } \beta_\Lambda \ll 1 \qquad (1\text{–}150)$$

and

$$\left.\begin{aligned} E_{\Lambda,\,\max} &= 16.96 E_\pi \\ E_{\Lambda,\,\min} &= 4.08 E_\pi \end{aligned}\right\} \qquad \text{for } \beta_\pi \simeq \beta_\Lambda \simeq 1 \qquad (1\text{–}151)$$

Noting that

$$R_{\Lambda \to n + \pi^0} \simeq 0.331$$

and

$$\zeta_{\gamma,\,\Lambda \to n + \pi^0} = 2 \qquad (1\text{–}152)$$

we can reduce equation (1–10) to yield the source function

$$q_\Lambda(E_\gamma, \mathbf{r}) = 44.9 n(\mathbf{r}) \int dE_p\, I(E_p, \mathbf{r}) \int_{E_\gamma + m_\pi^2/4E_\gamma}^{\infty} \frac{dE_\pi}{(E_\pi^2 - m_\pi^2)^{1/2}} \times \int_{(10.52 - 6.44\beta_\pi)E_\pi}^{(10.52 + 6.44\beta_\pi)E_\pi} \frac{dE_\Lambda \sigma(E_\Lambda | E_p)}{(E_\Lambda^2 - m_\Lambda^2)^{1/2}} \qquad (1\text{–}153)$$

In equation (1–153) we have used the double Green's function reduction

$$f_{\Lambda \to n + 2\gamma}(E_\gamma | E_\Lambda) = \int dE_\pi f_{\pi^0 \to 2\gamma}(E_\gamma | E_\pi) f_{\Lambda \to n + \pi^0}(E_\pi | E_\Lambda) \qquad (1\text{–}154)$$

1-5f The Decay Mode $\Sigma^+ \to p + \pi^0$ for Unpolarized Σ^+ Particles

The kinematics of this decay are exactly the same as those for the $\Lambda \to n + \pi^0$ decay; only the numerical values for the masses involved

are different. For this decay we have

$$m_{\Sigma^+} \simeq 1189.4 \text{ MeV}$$

and

$$m_p \simeq 938.26 \text{ MeV} \tag{1-155}$$

so that

$$\eta \simeq 0.195$$

$$p' \simeq 189.03 \text{ MeV}/c$$

and

$$\kappa \simeq 0.318$$

We also have, for this decay,

$$R_{\Sigma^+ \to p+\pi^0} \simeq 0.510$$

and

$$\zeta_{\gamma,\, \Sigma^+ \to p+\pi^0} \approx 2 \tag{1-156}$$

so that

$$q_{\Sigma^+}(E_\gamma, \mathbf{r}) = 40.3n(\mathbf{r}) \int dE_p\, I(E_p, \mathbf{r}) \int_{E_\gamma + (m_\pi/4E_\gamma)}^{\infty} \frac{dE_\pi}{(E_\pi^2 - m_\pi^2)^{1/2}}$$

$$\times \int_{(15.32-12.50\beta_\pi)E_\pi}^{(15.32+12.50\beta_\pi)E_\pi} \frac{dE_{\Sigma^+}\sigma(E_{\Sigma^+}|E_p)}{(E_{\Sigma^+}^2 - m_{\Sigma^+}^2)^{1/2}} \tag{1-157}$$

where

$$E_{\Sigma^+,\,\min} = 5.13E_\pi \qquad \text{for } \beta_{\Sigma^+} \ll 1 \tag{1-158}$$

and

$$E_{\Sigma^+,\,\min} = 2.82E_\pi \qquad \text{for } \beta_{\Sigma^+} \simeq 1 \tag{1-159}$$

1-5g The Three-Body Decay Modes $K_2^0 \to 3\pi^0$ and $K_2^0 \to \pi^+ + \pi^- + \pi^0$

We will treat these three-body decay modes by using the approximation $m_{\pi^\pm} \simeq m_{\pi^0} \equiv m_\pi$. We will also make use of the empirical fact that the K_2^0 mode is that of a spinless particle that decays isotropically into an S state.

The energy distribution function for the decay is given implicitly by

$$w = \frac{2\pi}{\hbar} M^2\, d^3 \left(\frac{dN}{dQ}\right) \tag{1-160}$$

where w is the decay rate into the three-differential energy region specified by the energies of the three particles formed in the decay, and dN/dQ is the density, in energy space, of the final states of the decay, with each degenerate state counted separately.

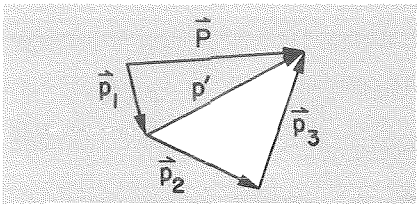

FIGURE 1–5.—The momentum vectors used in the derivation of equation (1–176).

The quantity M is the matrix element for the transition from the K_2^0 state to the 3π state and is of the form

$$M = \langle 3\pi | K_2^0 \rangle = (\text{spin and parity part}) \int d\tau_1 \, d\tau_2 \, d\tau_3 e^{i\epsilon_1 t} \, e^{i\epsilon_2 t} \, e^{i\epsilon_3 t} \cdots \propto (\epsilon_1 \epsilon_2 \epsilon_3)^{-1/2} \qquad (1\text{–}161)$$

where $d\tau$ represents an element of space-time volume, the subscripts denote the three decay pions, and the ϵ's denote the energies of each of the three pions. Thus,

$$M^2 \propto \frac{1}{\epsilon_1 \epsilon_2 \epsilon_3} \qquad (1\text{–}162)$$

We define

$$E = \epsilon_1 + \epsilon_2 + \epsilon_3 \qquad (1\text{–}163)$$

and

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \qquad (1\text{–}164)$$

in order to derive the phase-space part of equation (1–160). The derivation given here is essentially that given by Williams (1961). First, $\mathbf{p}_1$ is specified, leaving $\mathbf{p}'$ and ϵ' to be distributed among particles 2 and 3 as shown in figure 1–5. Then the phase-space volume for particles 2 and 3 is given by

$$U' = \int d\Omega_2 \int_0^{p_2(\theta_2)} p_2^2 \, dp_2 \qquad (1\text{–}165)$$

where $p_2(\theta_2)$ is the maximum momentum of particle 2 in direction θ_2. The remaining integration over particle 1 is given by

$$U = \int d\Omega_1 \int_0^{p_1(\theta_1)} p_1^2 \, dp_1 U' \qquad (1\text{–}166)$$

so that

$$N = \frac{V^2}{h^6} U = \frac{V^2}{h^6} \int d\Omega_1 \int_0^{p_1(\theta_1)} p_1^2 \, dp_1 \int d\Omega_2 \int_0^{p_2(\theta_2)} p_2^2 \, dp_2 \qquad (1\text{–}167)$$

and

$$d^3\left(\frac{dN}{dQ}\right) = d^3\left(\frac{dN}{dE}\right)_Q = \frac{d}{dE}\left\{\frac{V^2}{\hbar^6}\, d\Omega_1\, d\Omega_2\, p_1^2\, dp_1 \int_0^{p_2(\theta_2)} p_2^2\, dp_2\right\} \tag{1-168}$$

Letting p_2 be the independent momentum variable, we can write

$$\frac{dN}{dE} = \frac{dN}{dp_2}\frac{dp_2}{dE} \tag{1-169}$$

where

$$\frac{dE}{dp_2} = \frac{d\epsilon_1}{dp_2} + \frac{d\epsilon_2}{dp_2} + \frac{d\epsilon_3}{dp_2} \tag{1-170}$$

Now, ϵ_1 is fixed, so $d\epsilon_1/dp_2 = 0$. We also note that

$$\epsilon_{2,3}^2 = p_{2,3}^2 + m_\pi^2 \tag{1-171}$$

so that

$$\epsilon_{2,3}\, d\epsilon_{2,3} = p_{2,3}\, dp_{2,3} \tag{1-172}$$

or

$$\frac{d\epsilon_{2,3}}{dp_{2,3}} = \frac{p_{2,3}}{\epsilon_{2,3}} \tag{1-173}$$

Thus, from equations (1-170) and (1-173) we find

$$\frac{dE}{dp_2} = \frac{p_2}{\epsilon_2} + \frac{p_3}{\epsilon_3}\frac{dp_3}{dp_2} \tag{1-174}$$

Since $\mathbf{p}_1$ is fixed, and since fixing θ_2 merely specifies a coordinate axis, we have

$$\left(\frac{dp_3}{dp_2}\right)_{p_1} = -\cos\,\theta_{2,3} = -\frac{\mathbf{p}_2 \cdot \mathbf{p}_3}{p_2 p_3} \tag{1-175}$$

whereupon, by combining equations (1-168), (1-169), (1-174), and (1-175), we obtain the result

$$d^3\left(\frac{dN}{dE}\right) = \frac{V^2}{\hbar^6}\frac{\epsilon_2\epsilon_3 p_1^2 p_2^3\, d\Omega_1\, d\Omega_2\, dp_1}{p_2^2(E-\epsilon_1) - \epsilon_2 \mathbf{p}_2 \cdot (\mathbf{P}-\mathbf{p}_1)} \tag{1-176}$$

Referring to figure 1-6, we can write

$$p_3^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\,\theta \tag{1-177}$$

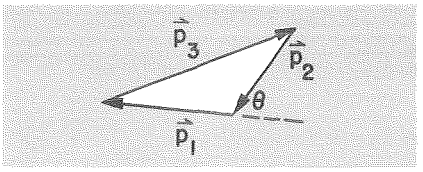

FIGURE 1–6.—The momentum vectors used in the derivation of equation (1–184).

so that

$$\mathbf{p}_2 \cdot (\mathbf{P} - \mathbf{p}_1) = \mathbf{p}_2 \cdot (\mathbf{p}_2 + \mathbf{p}_3) = p_2^2 + \mathbf{p}_2 \cdot \mathbf{p}_3 = \tfrac{1}{2}(p_1^2 - p_2^2 - p_3^2) + p_2^2 = p_1 p_2 \cos\theta \qquad (1\text{–}178)$$

Substituting equation (1–178) into equation (1–176), we obtain

$$d^3\left(\frac{dN}{dE}\right) = \frac{V^2}{\hbar^6} \frac{\epsilon_1 \epsilon_2 \epsilon_3 p_1 p_2^3 \, d\Omega_1 \, d\Omega_2 \, d\epsilon_1}{p_2^2(\epsilon_2 + \epsilon_3) + \epsilon_2 p_1 p_2 \cos\theta} \qquad (1\text{–}179)$$

The integration over the arbitrary $d\Omega_1$ yields 4π. We can write $d\Omega_2$ in the form

$$d\Omega_2 = 2\pi \, d(\cos\theta) \qquad (1\text{–}180)$$

For a specified ϵ_1, we have

$$d\epsilon_2 + d\epsilon_3 = 0 \qquad (1\text{–}181)$$

Using equation (1–181) in combination with equation (1–177), we obtain

$$-\epsilon_3 \, d\epsilon_2 = \epsilon_2 \, d\epsilon_2 + \frac{p_1 \epsilon_2}{p_2} \cos\theta \, d\epsilon_2 + p_1 p_2 \, d(\cos\theta) \qquad (1\text{–}182)$$

Thus,

$$-p_1 p_2^2 \, d(\cos\theta) = [p_2(\epsilon_1 + \epsilon_3) + \epsilon_2 p_1 \cos\theta] \, d\epsilon_2 \qquad (1\text{–}183)$$

whereupon equation (1–179) reduces to

$$d^2\left(\frac{dN}{dE}\right) = \frac{8\pi^2 V^2}{h^6} \epsilon_1 \epsilon_2 \epsilon_3 \, d\epsilon_1 \, d\epsilon_2 \qquad (1\text{–}184)$$

with the use of equation (1–183).

Substituting equations (1–184) and (1–163) into equation (1–160), we find

$$w \propto d\epsilon_1 \, d\epsilon_2 \propto dt_1 \, dt_2 \qquad (1\text{–}185)$$

where t_1 and t_2 are the kinetic energies of particles 1 and 2, respectively.

Because of the three-way symmetry of the kinetic relations in the problem of the decay into three particles of equal (or approximately equal) mass, equation (1–185) can be better expressed in the form

$$w \propto dt_i\, dt_k \qquad (i \neq k,\ i, k = 1, 2, 3) \tag{1–186}$$

This fact prompted Dalitz (1953) to introduce a plot of particle decays into a triangular energy space as shown in figure 1–7.

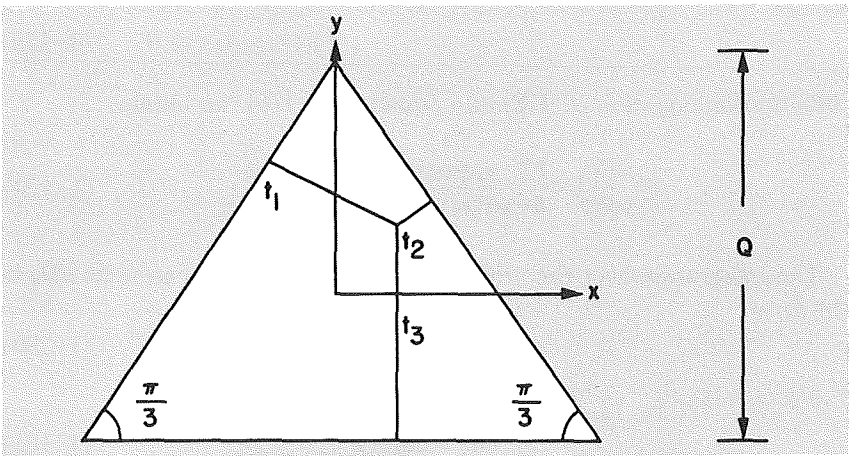

FIGURE 1–7.—The Dalitz triangle.

In this plot, the perpendicular distances from each side are proportional to the kinetic energies t_i of each of the decaying particles. The geometry of the triangle insures that $\Sigma t_i = Q$, Q being the total kinetic energy, $m_K - 3m_\pi$, released in the decay. From figure 1–7, we can see that

$$x = \frac{t_2 - t_1}{3^{1/2} Q}$$

and

$$y = \frac{Q - t_3}{3Q} \tag{1–187}$$

in units normalized to Q, so that a uniformity in $dt_i\, dt_k$ corresponds to a uniformity in $dx\, dy$. Thus, the uniform distribution of the decay points on the Dalitz plot, which is obtained experimentally, indicates a constancy in the matrix element M of equation (1–160) for the case of K_2^0 decay.

A more symmetric coordinate system for the Dalitz plot is obtained

by transformation to polar coordinates. In the polar coordinate system (ρ, ϕ)

$$\epsilon_1 = \tfrac{1}{3} m_K [1 + A\rho \cos(\phi - \tfrac{2}{3}\pi)]$$

$$\epsilon_2 = \tfrac{1}{3} m_K [1 + A\rho \cos(\phi + \tfrac{2}{3}\pi)] \qquad (1\text{–}188)$$

and

$$\epsilon_3 = \tfrac{1}{3} m_K (1 + A\rho \cos\phi)$$

where $A = Q/m_K$ (Fabri, 1954).

The condition of conservation of momentum further restricts the allowed area in the Dalitz plot, since we have the triangular relations

$$p_1 \leqslant p_2 + p_3$$

$$p_2 \leqslant p_1 + p_3 \qquad (1\text{–}189)$$

and

$$p_3 \leqslant p_1 + p_2$$

where

$$p_{1,2,3}^2 = \epsilon_{1,2,3}^2 - m_\pi^2 \qquad (1\text{–}190)$$

The limiting case is specified by the collinear distribution of momenta, where the equalities hold. From equations (1–188), (1–189), and (1–190), it is found that the limiting curve is given by

$$\rho_{\max}^2 = \frac{\frac{Q^2}{9}}{1 + B(1 + \rho \cos 3\phi)} \qquad (1\text{–}191)$$

so that all points on the Dalitz plot with $\rho \leqslant \rho_{\max}$ are allowed. In equation (1–191)

$$B = \frac{\frac{1}{2}A}{(1 - \frac{1}{2}A)^2} \qquad (1\text{–}192)$$

The nonrelativistic limit of the limiting curve is obtained in the event that $A \to 0$, which implies $B \to 0$ and $\rho_{\max} \to Q/3$. Thus, the nonrelativistic limiting curve is a circle, as shown in figure 1–8.

Equation (1–191) indicates only a slight distortion of the circle when relativistic effects are taken into account, since

$$A \simeq 0.19 \qquad B \simeq 0.11 \qquad \text{for } K_2^0 \to 3\pi^0 \text{ decay}$$

and (1–193)

$$A \simeq 0.17 \qquad B \simeq 0.10 \qquad \text{for } K_2^0 \to \pi^+ + \pi^- + \pi^0 \text{ decay}$$

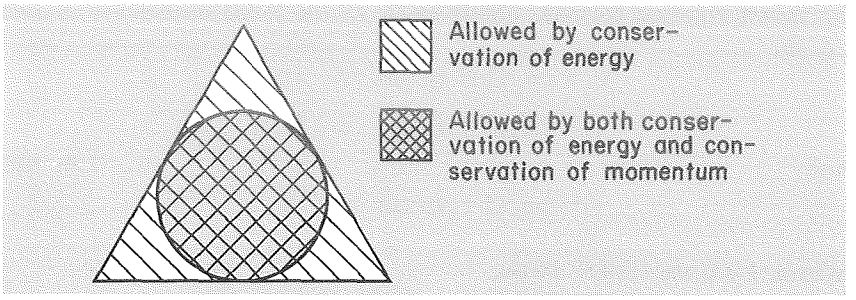

FIGURE 1–8.—Kinematic boundaries on the Dalitz triangle.

This small distortion will be further obliterated when the energy spectrum is spread out by transformation from the cms of the decay to the ls. Thus, we will find the nonrelativistic approximation quite adequate.

Referring to figure 1–9, we wish to find the energy distribution function $\Phi(\epsilon)$ for one of the particles; the results for the other two particles will be identical from symmetry considerations. Denoting this energy (in the cms of the kaon) as ϵ, we see from figure 1–9 that the probability for decay into the energy region between ϵ and $\epsilon + d\epsilon$ is proportional to the area in the rectangle. (There is uniformity within the circular area.) Thus,

$$\Phi(\epsilon)\ d\epsilon = K\lambda(\epsilon)\ d\epsilon \tag{1–194}$$

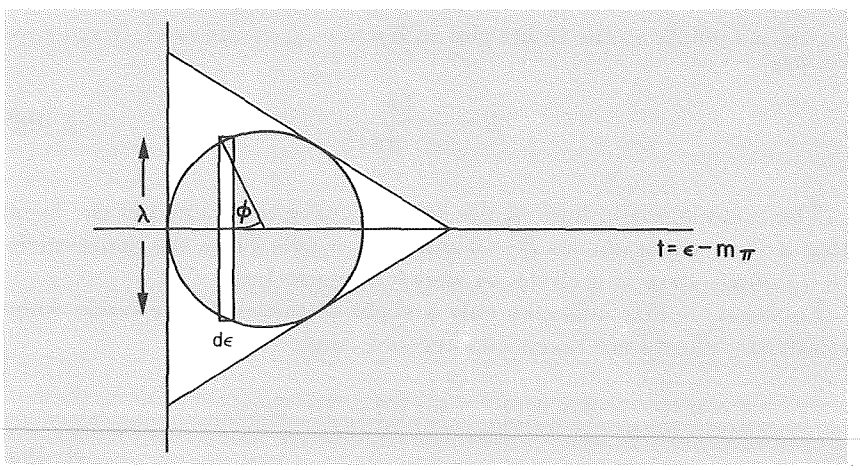

FIGURE 1–9.—The Dalitz diagram used in the derivation of equation (1–201).

where K is a normalization constant, such that

$$\int_{m_\pi}^{\epsilon_{max}} \Phi(\epsilon)\, d\epsilon = 1 = K \int_{m_\pi}^{\epsilon_{max}} \lambda(\epsilon)\, d\epsilon \qquad (1\text{–}195)$$

Referring to figure 1–9, we express λ and ϵ in terms of ϕ as

$$\lambda = \tfrac{2}{3} Q \sin \phi \qquad (1\text{–}196)$$

and

$$\epsilon - m_\pi = \tfrac{1}{3} Q (1 - \cos \phi) \qquad d\epsilon = \tfrac{1}{3} Q \sin \phi\, d\phi \qquad (1\text{–}197)$$

so that

$$K = \int_{m_\pi}^{\epsilon_{max}} \lambda\, d\epsilon = \int_0^\pi \frac{2Q^2}{9} \sin^2 \phi\, d\phi = \frac{Q^2}{9} \int_0^{2\pi} (1 - \cos^2 \phi)\, d\phi$$

$$= \pi \left(\frac{Q}{3}\right)^2 \qquad (1\text{–}198)$$

From equations (1–194) and (1–198), we find

$$\Phi(\epsilon)\, d\epsilon = \frac{1}{\pi (\frac{1}{3} Q)^2} \lambda(\epsilon)\, d\epsilon \qquad (1\text{–}199)$$

and, eliminating ϕ from equation (1–196) by using equation (1–197), we obtain

$$\lambda(\epsilon) = \frac{2}{3} Q \sin \left\{ 2 \left[\sin^{-1} \left(\frac{3}{2} \frac{\epsilon - m_\pi}{Q} \right)^{1/2} \right] \right\} \qquad (1\text{–}200)$$

so that

$$\Phi(\epsilon)\, d\epsilon = \frac{6}{\pi Q} \sin \left\{ 2 \left[\sin^{-1} \left(\frac{3}{2} \frac{\epsilon - m_\pi}{Q} \right)^{1/2} \right] \right\} d\epsilon \qquad (1\text{–}201)$$

Figure 1–10 shows $\Phi(\epsilon)$.

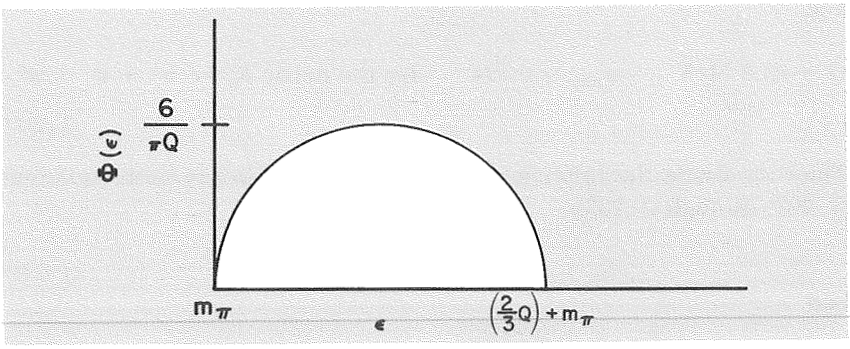

FIGURE 1–10.—The cms energy distribution function for pions formed in K_2^0 decay.

In the transformation to the ls,

$$E_\pi = \gamma_K(\epsilon + \beta_K p \cos\theta) \tag{1-202}$$

Thus

$$E_{\pi,\max} = \gamma_K\left(\frac{2}{3}Q + m_\pi + \Delta_{K,\max}\right) \tag{1-203}$$

where

$$\Delta_{K,\max} = \kappa(\epsilon_{\max})\,\beta_K(\tfrac{1}{2}E_K) \tag{1-204}$$

and

$$\kappa(\epsilon_{\max}) = \kappa_{\max} = 2\left[\frac{\left(\frac{2}{3}Q + m_\pi\right)^2 - m_\pi^2}{m_K^2}\right]^{1/2} \tag{1-205}$$

Thus

$$E_{\pi,\max} = E_K\left(\frac{m_\pi + \frac{2}{3}Q}{m_K} + \frac{1}{2}\beta_K\kappa_{\max}\right) \tag{1-206}$$

Again, we wish to determine the energy range of K_2^0-mesons that produce pions of energy E_π. In this case, however, we must take account of the finite range of decay energies E instead of the unique two-body decay energies we have previously considered. Setting $\eta \equiv \epsilon/m_K$, we again find

$$E_K = \left(\frac{m_K}{m_\pi}\right)^2 [\eta + \tfrac{1}{2}\beta_\pi\kappa\cos\theta']E_\pi \tag{1-207}$$

but in this case, η and κ are not constants but are functions of ϵ. The quantities Q and $\kappa_{\max}$ are given by

$$Q \simeq 93.0 \text{ MeV} \qquad \kappa_{\max} \simeq 0.560 \qquad \text{for the decay } K_2^0 \to 3\pi^0 \tag{1-208}$$

and

$$Q \simeq 83.8 \text{ MeV} \qquad \kappa_{\max} \simeq 0.534 \qquad \text{for the decay } K_2^0 \to \pi^+ + \pi^- + \pi^0 \tag{1-209}$$

Thus we obtain the limiting values for $K_2^0 \to 3\pi^0$ decay from equations (1–203) through (1–205):

$$E_K \simeq 2.53E_\pi \qquad \text{for } \beta_K \ll 1 \tag{1-210}$$

and

$$\left.\begin{aligned} E_{K,\max} &\simeq 9.27E_\pi \\ E_{K,\min} &\simeq 1.59E_\pi \end{aligned}\right\} \qquad \text{for } \beta_K \simeq \beta \simeq 1 \tag{1-211}$$

For $K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0$ decay

$$E_K \simeq 2.56 E_\pi \qquad \text{for } \beta_K \ll 1 \tag{1-212}$$

and

$$\left.\begin{aligned} E_{K,\max} &\simeq 8.96 E_\pi \\ E_{K,\min} &\simeq 1.64 E_\pi \end{aligned}\right\} \qquad \text{for } \beta_K \simeq \beta_\pi \simeq 1 \tag{1-213}$$

The appropriate constants for the two decay modes of K_2^0 considered are

$$\zeta = 6 \qquad R \simeq 0.265 \qquad \text{for } K_2^0 \rightarrow 3\pi^0 \tag{1-214}$$

and

$$\zeta = 2 \qquad R \simeq 0.114 \qquad \text{for } K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0 \tag{1-215}$$

We will thus neglect the small kinematic differences between the $K_2^0 \rightarrow 3\pi^0$ and $K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0$ decay modes and use the quantities derived for the $K_2^0 \rightarrow 3\pi^0$ mode, which contributes more than six times as many γ-rays as does the $K_2^0 \rightarrow \pi^+ + \pi^- + \pi^0$ mode.

We also note that

$$\frac{\sigma(E_{K_2^0}|E_p)}{\sigma(E_{K^0}|E_p)} = \frac{1}{2} \tag{1-216}$$

This, together with equations (1–10) through (1–14), (1–99), (1–207), (1–214), and (1–215), yields the source spectrum

$$q_{K_2^0}(E_\gamma, \mathbf{r}) = 11.43 n(\mathbf{r}) \int dE_p I(E_p, \mathbf{r}) \int_{E_\gamma + (m_\pi^2/4E_\gamma)}^{\infty} \frac{dE_\pi}{(E_\pi^2 - m_\pi^2)^{1/2}}$$
$$\times \int_{m_\pi}^{m_\pi + 2/3Q} \frac{d\epsilon \Phi(\epsilon)}{\kappa(\epsilon)} \int_{\left(\frac{m_K}{m_\pi}\right)^2 [\eta(\epsilon) - \beta_\pi \kappa(\epsilon)/2] E_\pi}^{\left(\frac{m_K}{m_\pi}\right)^2 [\eta(\epsilon) + \beta_\pi \kappa(\epsilon)/2] E_\pi} \frac{dE_K \sigma(E_K|E_p)}{(E_K^2 - m_K^2)^{1/2}} \tag{1-217}$$

1-6 SUMMARY OF VARIOUS KINEMATIC PROPERTIES OF THE IMPORTANT GAMMA-RAY-PRODUCING DECAY MODES

In our discussion in the previous section, we established the Lorentz transformation equation for the two-body decay process given by equation (1–69). When we evaluate the cosmic γ-ray production spectrum later on, we will find it useful to give our results in the form of a log-log plot covering the many decades in energy which are of astrophysical significance. At this point, before we summarize the kinematic properties described in the previous section, we digress a bit to point out an addi-

tional symmetry property which holds for logarithmic energy spectra that is implicit in equation (1–69) for two-body decays.

From equation (1–69), it follows that a particle which is the product of a two-body decay will have an ls energy which lies between the minimum and maximum values obtained when $\cos\theta' = \pm 1$. These values are

$$E_{s,\,\min} = E_{\rm pr}(\eta - \beta\rho) \tag{1–218}$$

and

$$E_{s,\,\max} = E_{\rm pr}(\eta + \beta\rho) \tag{1–219}$$

where $E_{\rm pr}$ and E_s are the energies of the primary and secondary particles, respectively. We have replaced the quantity κ in equation (1–69) by ρ to avoid confusion with the definition of κ in the previous section. The product

$$(E_{s,\,\min})(E_{s,\,\max}) = (\eta^2 - \beta^2\rho^2)E_{\rm pr}^2 \tag{1–220}$$

We now define a geometric mean for the energy range of the secondary particle

$$E_s = E_{s,\,\min}E_{s,\,\max} = (\eta^2 - \beta^2\rho^2)^{1/2}E_{\rm pr} \tag{1–221}$$

Thus, in the nonrelativistic limit,

$$E_s \simeq \eta E_{\rm pr} \simeq \mu \qquad (\gamma \simeq 1,\ \beta \ll 1) \tag{1–222}$$

where μ, as before, is the energy of the secondary in the rest system of the decaying primary.

In the relativistic limit,

$$\tilde{E}_s \simeq (\eta^2 - \rho^2)^{1/2}E_{\rm pr} = (M_s/M_{\rm pr})E_{\rm pr} = \gamma M_s \qquad (\beta \simeq 1) \tag{1–223}$$

In particular, for photon secondaries

$$\tilde{E}_\gamma = \eta(1 - \beta^2)^{1/2}E_{\rm pr} = \mu \tag{1–224}$$

We have thus proved an important kinematic property regarding the energy range of secondary γ-rays that are the product of two-body decays; viz,

> The geometric mean of the energy range of secondary γ-rays which are produced in all two-body decays is equal to the energy of the γ-rays in the rest system of the decaying primary μ and is independent of the energy of the primary particle.

Also, since

$$\log E_\gamma = \tfrac{1}{2}(\log E_{\gamma,\,\min} + \log E_{\gamma,\,\max}) = \log \mu \qquad (1\text{–}225)$$

it follows that, on logarithmic plots of the energy spectra of these γ-rays, the rest-system energy μ will lie halfway between the extremum energies.

We are particularly concerned with decays that are isotropic in the rest system of the decaying particle, such as the π^0 and Σ^0 decays, which we have previously considered. For these decays, we have already shown that the resultant γ-ray energy distribution function is only a function of the momentum of the primary; indeed this function is a constant which is inversely proportional to this momentum for a given primary, within a range proportional to the momentum of the primary, and vanishes outside this range. Thus, for decays of parent particles with a wide range of primary energies, γ-ray spectra are generated which are made up of a superposition of rectangular spectra, as shown in figure 1–11. Higher energy primaries produce the γ-rays at the extremes of the spectrum. We therefore deduce a second important kinematic property, which holds for two-body decays that produce γ-rays isotropically in the rest system of the decaying primary; viz,

> The energy spectra of γ-rays produced isotropically in the rest system of the decaying primary will be symmetric on a logarithmic plot with respect to $E_\gamma = \mu$ and will peak at $E_\gamma = \mu$.

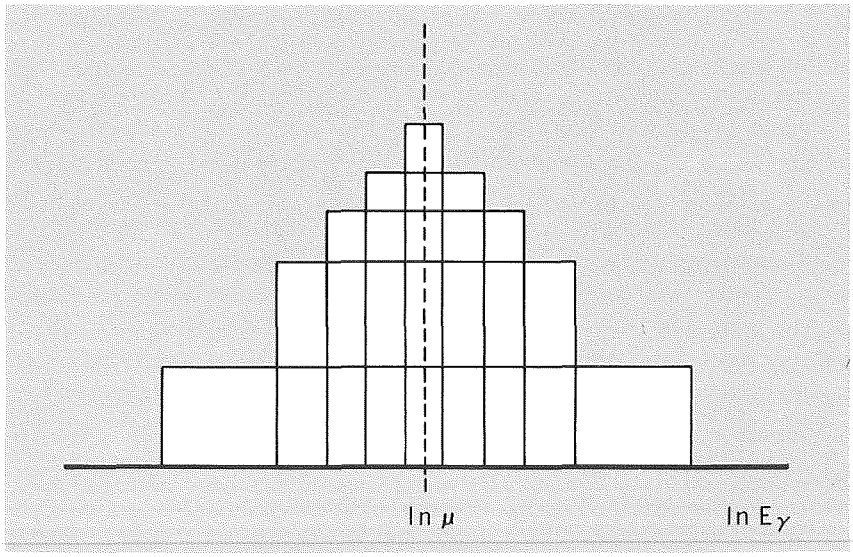

FIGURE 1–11.—Ideal superposition of γ-ray energy spectra from π^0 of Σ^0 particles having discrete values of energy.

The properties of the various decay modes discussed in the previous section are given in table 1–3.

TABLE 1–3.—*Kinematic Properties of γ-Ray-Producing Decay Modes*

Decay mode ($1 \rightarrow 2$)	$E_1(\beta_1 \ll 1)$	$E_{1,\min}(\beta_1 \simeq 1)$	$E_{1,\max}(\beta_1 \simeq 1)$	$\tilde{E}_2(\beta_1 \ll 1)$	$\tilde{E}_2(\beta_1 \simeq 1)$
$\pi^0 \rightarrow 2\gamma$	$2E_\gamma$	E_γ	∞	$M_\pi/2$	$M_\pi/2$
$\Sigma^0 \rightarrow \Lambda + \gamma$	E_γ/η	$E_\gamma/2\eta$	∞	$\frac{M_\Sigma^2 - M_\Lambda^2}{2M_\Sigma}$	$\frac{M_\Sigma^2 - M_\Lambda^2}{2M_\Sigma}$
$K_1^0 \rightarrow 2\pi^0$	$2E_\pi$	$1.11E_\pi$	$12.7E_\pi$	$M_K/2$	$0.272E_K$
$K^\pm \rightarrow \pi^\pm + \pi^0$	$2.01E_\pi$	$1.09E_\pi$	$12.5E_\pi$	$0.498M_K$	$0.274E_K$
$\Lambda \rightarrow n + \pi^0$	$6.58E_\pi$	$4.08E_\pi$	$16.96E_\pi$	$0.152E_\Lambda$	$0.121E_\Lambda$
$\Sigma^+ \rightarrow p + \pi^0$	$5.13E_\pi$	$2.82E_\pi$	$27.82E_\pi$	$0.195E_\Sigma$	$0.113E_\Sigma$
$K_2^0 \rightarrow 3\pi^0$	$2.53E_\pi$	$1.59E_\pi$	$9.27E_\pi$		
$K^0 \rightarrow \pi^+ + \pi^- + \pi^0$	$2.56E_\pi$	$1.64E_\pi$	$8.96E_\pi$		